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On hydromagnetic waves in a stratified rotating incompressible fluid

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The dispersion relationship for plane hydromagnetic waves in a stratified rotating fluid (*a*) indicates that the well-known analogy between rotating fluids and stratified fluids in regard to their hydrodynamic behaviour does not extend to magnetohydrodynamic behaviour, and (*b*) lends credence to a certain conjecture made in a previous paper, namely that effects due to density stratification can be neglected when considering the dispersion relationship for free hydromagnetic oscillations of the Earth's core if the Brunt–Väisälä frequency is much less than twice the angular speed of the Earth's rotation.

1. Introduction

In a previous paper (Hide 1966) an approximate dispersion relationship for free hydromagnetic oscillations of the fluid core of the Earth was proposed, and it was conjectured that when

$$|\mathbf{N}|^2 \ll 4|\mathbf{\Omega}|^2 \tag{1.1}$$

effects due to vertical gradients of density, ρ, can safely be ignored. Here $\mathbf{\Omega}$ is the angular velocity of the Earth's rotation and

$$\mathbf{N} \equiv \frac{\mathbf{g}}{g}\left(\frac{g}{\rho}\frac{\partial \rho}{\partial z}\right)^{\frac{1}{2}}; \tag{1.2}$$

$|\mathbf{N}|$ is the Brunt–Väisälä frequency (effects due to compressibility being negligible in the theoretical model) and $\mathbf{g} = (0, 0, g)$ is the acceleration due to gravity and centrifugal effects. The purpose of the present note is to investigate the validity of the criterion expressed by equation (1.1).

The most acceptable procedure would be to find an accurate dispersion relationship for the case $\mathbf{N} \neq 0$, but this has not yet proved feasible. Owing to the (nearly) spherical geometry of the system, even when $\mathbf{N} = 0$ the problem is mathematically intractable (see Hide 1966; Malkus 1967; Stewartson 1967; Suffolk & Allan 1969), except in certain very special cases. For this reason we shall examine an elementary but related problem whose solution should indicate, in part at least, how effects due to rotation, density stratification and magnetic fields interact with one another. Thus, we shall consider plane, small amplitude, harmonic waves propagating in an inviscid, perfectly conducting, incompressible, rotating fluid of indefinite extent in all directions when both $\mathbf{N}$ (based on the undisturbed density field $\rho = \rho_0(z)$) and

$$\mathbf{V} \equiv \mathbf{B}_0/\sqrt{(\mu\rho_0)}, \tag{1.3}$$

the Alfvén velocity, are uniform, where $\mathbf{B}_0$ is the undisturbed magnetic field vector and μ is the magnetic permeability.

2. Equations of the problem

The equations of the problem referred to a frame which rotates with steady angular velocity $\mathbf{\Omega}$ relative to an inertial frame are, when all transport processes (viscosity, electrical resistivity, thermal conduction, etc.) are negligible, the following:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\,\mathbf{u} + 2\mathbf{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{\mathbf{j}\times\mathbf{B}}{\rho} + \mathbf{g}, \tag{2.1}$$

$$\nabla.\mathbf{u} = 0, \tag{2.2}$$

$$\frac{\partial \rho}{\partial t} + (\mathbf{u}.\nabla)\rho = 0, \tag{2.3}$$

$$\nabla.\mathbf{B} = 0, \tag{2.4}$$

$$\nabla\times\mathbf{B} = \mu\mathbf{j}, \tag{2.5}$$

$$\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t, \tag{2.6}$$

$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = 0. \tag{2.7}$$

Here $\mathbf{u}$ denotes the Eulerian flow velocity, p pressure, $\mathbf{j}$ current density, $\mathbf{E}$ electric field, and t time. Equations (2.1)–(2.3) express conservation of momentum, matter and density of individual fluid elements, respectively; equations (2.4)–(2.6) are the laws of Gauss, Ampère and Faraday, respectively; equation (2.7) states that in a perfect conductor of electricity, the electric field acting on a moving element must vanish because otherwise, by Ohm's law, electric currents of infinite strength would be implied.

If we write

$$\left.\begin{aligned} \rho &= \rho_0(z) + \rho_1(x,y,z,t), & \mathbf{u} &= \mathbf{u}_0 + \mathbf{u}_1(x,y,z,t), \\ p &= p_0(z) + p_1(x,y,z,t), & \mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1(x,y,z,t), \end{aligned}\right\} \tag{2.8}$$

where $\mathbf{u}_0 = 0$ and $\nabla p_0 = \mathbf{g}\rho_0$, and we assume that $|p_1| \ll |p_0|$, $|\mathbf{B}_1| \ll |\mathbf{B}_0|$ $|\rho_1| \ll \rho_0$ and spatial variations in ρ_0 are small in comparison with ρ_0, then to first order of small quantities p_1, $\mathbf{B}_1$, ρ_1 and $\mathbf{u}_1 = (u_1, v_1, w_1)$ equations (2.1)–(2.4) become:

$$\frac{\partial \mathbf{u}_1}{\partial t} + 2\mathbf{\Omega}\times\mathbf{u}_1 = -\frac{1}{\bar{\rho}_0}\nabla p_1 + \frac{(\nabla\times\mathbf{B}_1)\times\mathbf{B}_0}{\mu\bar{\rho}_0} + \mathbf{g}\frac{\rho_1}{\rho_0} \tag{2.9}$$

(where $\bar{\rho}_0$ is the mean density of the fluid),

$$\nabla.\mathbf{u}_1 = 0, \tag{2.10}$$

$$\frac{\partial \rho_1}{\partial t} + w_1\frac{d\rho_0}{dz} = 0, \tag{2.11}$$

$$\nabla.\mathbf{B}_1 = 0 \tag{2.12}$$

—equation (2.5) having been used to eliminate $\mathbf{j}$ from (2.1)—and the equation that results when $\mathbf{E}$ is eliminated between (2.6) and (2.7) becomes:

$$\frac{\partial \mathbf{B}_1}{\partial t} - (\mathbf{B}_0 . \nabla)\mathbf{u}_1 = 0. \tag{2.13}$$

Eliminate p_1, $\mathbf{B}_1$ and ρ_1 between (2.9)–(2.13) and show that:

$$\left[\frac{\partial^2}{\partial t^2} - (\mathbf{V}.\nabla)^2\right]\nabla\times\mathbf{u}_1 - 2(\mathbf{\Omega}.\nabla)\frac{\partial \mathbf{u}_1}{\partial t} - \mathbf{N}\times\nabla(\mathbf{N}.\mathbf{u}) = 0 \tag{2.14}$$

(cf. (1.2) and (1.3)).

Substitute a plane wave solution,

$$\mathbf{u} \propto \exp\{i(\omega t - \boldsymbol{\kappa}.\mathbf{r})\} \tag{2.15}$$

(where ω is the angular frequency, $\boldsymbol{\kappa} = (k, l, m)$ is the wave-number vector and $\mathbf{r} = (x, y, z)$), in (2.10) and (2.14) and thus find, after a little manipulation, the dispersion relationship:

$$\omega^2 = (\mathbf{V}.\boldsymbol{\kappa})^2 + \frac{1}{2}\left\{\frac{(\mathbf{N}\times\boldsymbol{\kappa})^2 + (2\mathbf{\Omega}.\boldsymbol{\kappa})^2}{\kappa^2} \pm \left[\left(\frac{(\mathbf{N}\times\boldsymbol{\kappa})^2 + (2\mathbf{\Omega}.\boldsymbol{\kappa})^2}{\kappa^2}\right)^2 + \frac{4(\mathbf{V}.\boldsymbol{\kappa})^2(2\mathbf{\Omega}.\boldsymbol{\kappa})^2}{\kappa^2}\right]^{\frac{1}{2}}\right\}. \tag{2.16}$$

The phase velocity, $\omega\boldsymbol{\kappa}/\kappa^2$, and group velocity, $(\partial\omega/\partial k,\ \partial\omega/\partial l,\ \partial\omega/\partial m)$, follow directly from (2.16). When $N^2 \equiv \rho_0^{-1} g d\rho_0/dz > 0$, ω is always real; we restrict attention in what follows to this case of stable density stratification. (In the other case, $d\rho_0/dz < 0$, the density stratification is unstable, as evinced by the result that ω may then take complex values with negative imaginary parts.)

Particle displacements are transverse with respect to the wave fronts, i.e. $\mathbf{u}_1.\boldsymbol{\kappa} = 0$, the occurrence of non-zero values of $\mathbf{u}_1.\boldsymbol{\kappa}$ being incompatible with the assumption of incompressibility (see equation (2.10)). Details of the shapes of particle orbits (linear, circular or elliptical) and of concomitant disturbances of the magnetic field and of the fields of density and vorticity, are readily derived from (2.9)–(2.16).

3. Discussion

In three limiting cases, (2.16) reduces to particularly simple forms. Thus, when $\mathbf{V} = \mathbf{N} = 0$ we have 'inertial waves', for which

$$\omega^2 = (2\mathbf{\Omega}.\boldsymbol{\kappa})^2/\kappa^2 \tag{3.1}$$

(cf. Greenspan 1968), and in which rotation provides the restoring forces; when $\mathbf{V} = \mathbf{\Omega} = 0$ we have 'internal waves', for which

$$\omega^2 = (\mathbf{N}\times\boldsymbol{\kappa})^2/\kappa^2 \tag{3.2}$$

(cf. Yih 1965), and in which buoyancy effects provide the restoring forces; and when $\mathbf{\Omega} = \mathbf{N} = 0$ we have 'hydromagnetic (magnetohydrodynamic or Alfvén) waves', for which

$$\omega^2 = (\mathbf{V}.\boldsymbol{\kappa})^2 \tag{3.3}$$

(cf. Alfvén & Fälthammar 1963, or Roberts 1967), and in which the magnetic field provides the restoring forces. Inertial waves and internal waves are highly dispersive, having group velocities that depend on $\boldsymbol{\kappa}$. Alfvén waves, however, are non-dispersive; they propagate with group velocity equal to $\mathbf{V}$, which is independent of $\boldsymbol{\kappa}$.

It has occurred to a number of workers (see Greenspan 1968) that in some respects rotating fluids and stratified fluids exhibit analogous hydrodynamic behaviour (cf. equations (3.1) and (3.2), especially when $l = 0$ and $\boldsymbol{\Omega} = (\Omega, 0, 0)$), but (2.16) shows that the analogy does not extend to magnetohydrodynamic behaviour. Thus, when $\boldsymbol{\Omega} = 0$

$$\omega^2 = (\mathbf{V}.\boldsymbol{\kappa})^2 + \frac{1}{2}\left\{\frac{(\mathbf{N}\times\boldsymbol{\kappa})^2}{\kappa^2}\right\}[1 \pm 1] \tag{3.4}$$

(the negative sign corresponding to modes for which particle displacements are horizontal and therefore only the magnetic field provides the restoring force), whereas when $\mathbf{N} = 0$

$$\omega^2 = (\mathbf{V}.\boldsymbol{\kappa})^2 + \frac{1}{2}\left\{\frac{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2}{\kappa^2}\right\}\left[1 \pm \left(1 + \frac{4(\mathbf{V}.\boldsymbol{\kappa})^2\kappa^2}{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2}\right)^{\frac{1}{2}}\right] \tag{3.5}$$

(cf. Lehnert 1954; Hide 1955; Chandrasekhar 1961; Lighthill 1967). The important difference between the last two equations is the presence in the latter of a term containing the (square of the) ratio of $\mathbf{V}.\boldsymbol{\kappa}$ to $(2\boldsymbol{\Omega}.\boldsymbol{\kappa})/\kappa$, which has no counterpart in the former. (When $\mathbf{V} = 0$, $\omega^2 = (\frac{1}{2} \pm \frac{1}{2})\{(\mathbf{N}\times\boldsymbol{\kappa})^2 + (2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2\}/\kappa^2$.)

This 'ratio term', which arises because Coriolis forces act at right angles to $\mathbf{u}_1$ and thus prevent the occurrence of 'decoupled' modes, is of great physical importance, especially when $4(\mathbf{V}.\boldsymbol{\kappa})^2 \ll (2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2/\kappa^2$ (cf. equations (3.1) and (3.3)). The roots of equation (3.5) are then

$$\omega^2 \doteqdot \frac{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2}{\kappa^2} \quad \text{and} \quad \omega^2 \doteqdot \left[\frac{(\mathbf{V}.\boldsymbol{\kappa})^2\kappa}{(2\boldsymbol{\Omega}.\boldsymbol{\kappa})}\right]^2, \tag{3.6}$$

the latter being the dispersion relationship for a hybrid type of wave (see Lehnert 1954; Chandrasekhar 1961; Hide 1966), which has no direct analogue in a stratified fluid.

In conclusion, observe that when $|\mathbf{N}\times\boldsymbol{\kappa}| \ll |2\boldsymbol{\Omega}.\boldsymbol{\kappa}|$ (3.5) is a very good approximation to (2.16), indicating that the conjecture expressed by (1.1) is probably valid, at least for oscillations with wavelengths that are much smaller than the radius of the Earth's core.

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REFERENCES

Alfvén, H. & Fälthammar, C. G. 1963 *Cosmical Electrodynamics*, 2nd ed. Oxford University Press.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press.

GREENSPAN, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.

HIDE, R. 1955 *Proc. Roy. Soc. Lond.* A, **233**, 376–396.

HIDE, R. 1966 *Phil. Trans. Roy. Soc. Lond.* A, **259**, 615–647.

LEHNERT, B. 1954 *Astrophys. J.* **119**, 647–654.

LIGHTHILL, M. J. 1967 In *Aerodynamic Phenomena in Stellar Atmospheres* (ed. R. N. Thomas). New York: Academic Press.

MALKUS, W. V. R. 1967 *J. Fluid Mech.* **28**, 793–802.

ROBERTS, P. H. 1967 *An Introduction to Magnetohydrodynamics*. New York: American Elsevier; London: Longman.

STEWARTSON, K. 1967 *Proc. Roy. Soc. Lond.* A **299**, 173–187.

SUFFOLK, G. & ALLAN, D. W. 1969 *The Application of Modern Physics to the Earth and Planetary Interiors* (ed. S. K. Runcorn). London: Wiley—Interscience.

YIH, C. S. 1965 *Dynamics of Non-homogeneous Fluids*. New York: Macmillan.